take a bite out of FRACTION DIVISION

When it comes to fractions, students often understand just part of the story. Assign some meaningful problems to help them see the whole picture.

Nesrin Cengiz and Margaret Rathouz

"Division of fractions is often considered the most mechanical and least understood topic in elementary school" (Tirosh 2000, p. 6). Enacting fraction division tasks in meaningful ways requires that teachers know not only *how* fraction division works but also *why* it works (e.g., Cramer et al. 2010). We have created materials to help preservice teachers develop that knowledge. To provide sufficient time for the learners to engage and struggle with cognitively demanding tasks (Stein et al. 2009), just one or two complex problems are explored during one-hundred-minute periods. Various instructional strategies are used to elicit multiple solution methods and representations and to press for justifications of solutions, modeling the type of instruction these teachers could use with their future students (Grant, Lo, and Flowers 2007; Rathouz 2009).

A sequence of tasks will be presented that is designed to promote an understanding of fraction division, and brief vignettes will explore their implementation. In particular, following a discussion of prerequisite experiences in whole-number division, two types of fraction division activities will be addressed: those helping build the concept of the unit and those helping connect representations (see **fig. 1**). Although these tasks were implemented with preservice teachers, examining these tasks may help in-service teachers deepen their own understanding of fraction division. Further, the activities provide seeds for mathematical tasks that teachers might enact with middle school learners. We suggest that the reader work on each task as it arises in the article. Then, read about our experience implementing each activity, and reflect on how it might be modified for use with middle school students.

BUILDING THE CONCEPT OF THE UNIT

In initial tasks of the fractions unit, preservice teachers begin to recognize the roles of the numerator and the denominator and to appreciate the need to identify, label, and track units while solving fraction problems. During an in-class discussion of the task shown in **figure 2**, preservice teachers solidify these important ideas as they share their problem-solving processes and reasoning. The different approaches described below by Kelly and Jason highlight the relevance of the numerator, denominator, and various units. (All names are pseudonyms.)

> *Kelly*: I saw the 2/5 of a batch as 2 groups out of the 5 groups I'd need for one batch. So I divided the 6 cookies into 2 groups of 3 cookies. Then I multiplied the 3 cookies by 5 to get 15 cookies in a whole batch.
>
> *Jason*: I thought about it as "I have 2/5 of a batch." When I double it, I have 4/5 of a batch. I need one more fifth, so I cut the 6 cookies in half to get how many are in 1/5 of a batch. Then 5/5 is a whole batch.

This problem and the descriptions illustrate the various units that need to be tracked in this division problem (6   2/5), including "the cookie," "the batch," "the group of 3 cookies," and "the 1/5 batch." Although connections to fraction division are not made at this stage, eliciting multiple solutions and pressing for reasoning help learners make sense of the part-whole interpretation of a fraction and the role of the numerator (the number of groups shown) and denominator (the number of equal groups in the whole batch).

Fig. 1 Two story problems represented by the same division expression can elicit rich discussion.

Consider the following two stories for 12   3 = ?

1. Sally has 12 marbles, and she wants to give 3 to each friend. To how many friends can she give her marbles?
2. Sally has 12 marbles, and she wants to distribute them among three friends. How many marbles will she give to each friend?

Note that in the first problem (*measurement division*), the size of the group (3 marbles per friend) is known and the number of groups (4 friends) is unknown. In the second problem (*partitive division*), the number of groups (3 friends) to be formed is known and the size of the group (4 marbles per friend) is unknown. Although middle school students do not necessarily need to name these division situations, teachers should fully understand the two processes to ensure that students are exposed to both interpretations. Further, using each interpretation of division in fraction situations is critical for helping students make connections between the context and the method of calculation.

CONNECTING REPRESENTATIONS

Distinguishing among different operations on fractions is challenging for both young students and adults (Armstrong and Bezuk 1995; Tirosh 2000). Making connections among stories, diagrams, and symbols can help learners identify characteristics of each operation. The following tasks help build these connections for fraction division.

From Stories to Diagrams and Symbols

The task shown in **figure 3** requires identifying multiplication and division in story contexts and associating those contexts with numeric expressions.

The first story is a measurement division problem; the size of the "groups" is known, and the number of groups is needed. The emergence of several possible ways to name the remainder creates an opening to discuss the critical issue of the unit and how it shifts from "cups of flour" to "batches of cookies." An initial (erroneously labeled) diagram produced by preservice teachers to solve question 1 appears in **figure 4**. Their explanations follow:

> *Sara*: Half a cup makes 1 batch, and half of 6 is 3. So you have 2/6 left in your cup, which simplifies to 1/3. I made 1 batch here, so altogether I've made 1 1/3 batch.
>
> *Teacher*: Alright, so everybody think about it for a second. [Pause] Questions or comments?
>
> *Megan*: We don't need a whole cup. If we only need a half to make a full batch, then don't we only need to count to the next half, so you would have 2/3 of 1/2, which gives you 2/3 of a batch? [Means that Sara's 2/6 cup of flour corresponds to 2/3 of a batch]

Fig. 2 Posing this task helps students build the concept of the *unit.*

I baked a batch of happy-face cookies. You see 2/5 of the batch below. How many cookies were in the batch?

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Fig. 3 This task requires students to connect stories to symbols.

Draw diagrams and use them to solve each word problem:

1. Maura has 5/6 cup of flour. Each batch of cookies requires 1/2 cup flour. How many batches of cookies can she make?
2. Maura has 5/6 of a batch of cookies. She plans to share the cookies equally with her friend, Jamilla. How many batches will each of them get?

Determine which, if any, of the four given arithmetic expressions correspond to the problems above. This determination must be based on the meanings of the operations, *not* on the correctness of the answer. If you can justify the connection, you may list more than one arithmetic statement for each.

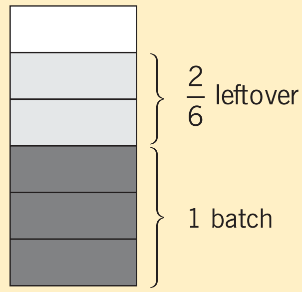
a. $\frac{5}{6} \div 2$ b. $\frac{5}{6} \div \frac{1}{2}$

c. $\frac{5}{6} \times 2$ or $2 \times \frac{5}{6}$ d. $\frac{5}{6} \times \frac{1}{2}$ or $\frac{1}{2} \times \frac{5}{6}$

Discuss connections between the problem situation, your diagram, and the arithmetic. Be sure you focus on the meaning of the operations and *keep track of the units* of the numbers being operated on and of the result of that operation.

Source: Adapted from Schifter, Bastable, and Russell 1999

Fig. 4 This illustration is an incomplete diagram for question 1 in **figure 3**.



Danielle: The 2/6 is measuring how much of a cup is leftover, and we want to know how much of a batch is leftover. Since we only have enough to make 2/3 of another batch, the answer should be 1 2/3 batches.

Analyzing work in progress provides an opportunity to interpret, revise, and complete solutions collaboratively. Encouraging learners to reflect on the shared diagram, while referring to the problem context and naming units for all numbers, allows them to convince one another that 1 2/3 batches makes more sense. As they continue to work on problems in **figure 3**, issues arise concerning fraction multiplication and division. In the second story problem, 5/6 batch is shared between 2 people (the number of groups is 2 and the size of each group is needed). Some recognize that it is partitive division and that the expression 5/6   2 goes with the problem. Others view the problem as finding 1/2 of 5/6 and choose

$$\frac{5}{6} \times \frac{1}{2} \text{or} \ \frac{1}{2} \times \frac{5}{6}$$

as corresponding expressions. They realize that the commutative property holds for multiplication. Discussing this story and these two expressions reinforces the idea that multiplying by a unit fraction (1/2) is equivalent to dividing by its whole-number reciprocal (2). This sets the groundwork for thinking about division by any fraction as multiplication by its reciprocal, as occurs in the standard invert-and-multiply algorithm.

From Symbols to Stories and Diagrams

In contrast to the previous task, the next task requires constructing stories to match given division calculations and creating diagrams to use in solving their problems. The calculation given was

$$2\frac{1}{2} \quad \frac{3}{4},$$

and students were required to provide two unique stories and diagrams using different interpretations of division. We ask preservice teachers to create stories and diagrams to illustrate their understandings of the two meanings of fraction division (partitive and measurement), so that they will be able to use both interpretations flexibly when working with their students.

We intentionally ask our preservice teachers to begin work independently for two reasons: (1) to allow them time to think; and (2) to promote diversity in stories and solutions, which will be shared with their small group and the class. One group's story and initial diagram are shown in **figure 5a**. The rest of the class is asked to reflect on what type of division (partitive or measurement) is represented by the story and how the diagram could be used to solve the problem.

It is not uncommon for "Why is this division?" and "What should we call the remainder?" to arise multiple times during this unit. These questions occurred with the group of students who are highlighted in

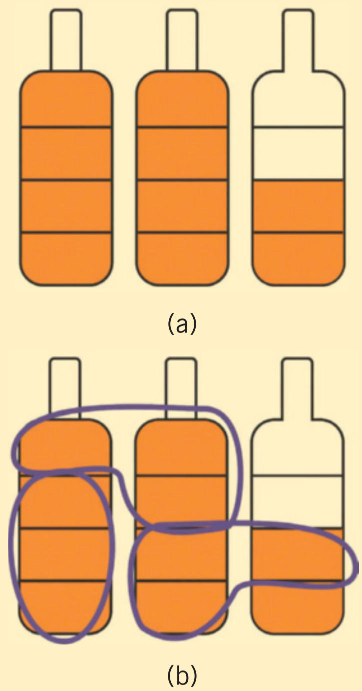
Stories and Diagrams

In the following dialogue, the teacher and students are discussing stories and diagrams that they used to represent 2 1/2   3/4.

Teacher: I asked this group to show its work in progress. Can you tell from the story and the diagram how they were thinking about division?
Melissa: Aren't they saying that 3/4 is the serving size, so they're dividing each gallon into fourths? So then she can count 1, 2, 3 [fourths]; that's one serving. There's one leftover fourth to start the next serving. Then keep counting out servings as 3/4 of a gallon.
David: But why is that division?
Teacher: I think several people are wondering that. So I'd like you to go back to your small groups and discuss why this might be a division problem. [Preservice teachers have a brief small-group discussion and return to the whole group.]
Melissa: I think it's like asking how many 3/4 gallon servings are in 2 1/2 gallons. Isn't that measurement division?
David: OK, I can see that . . . like with whole-number division, 12 divided by 3 means how many 3s are in 12.
Teacher: More of you seem convinced that this is a division problem. Now how are we going to use that diagram to solve the problem?
Taylor: She already separated them into fourths. I would just circle groups of 3 [see **fig. 5b**] like this. So three servings is what I see. But there's 1/4 leftover.
Nicole: Isn't that last fourth really 1/3 of a serving?
Teacher: Maybe we should label the diagram so we know what's what. There seems to be an issue around this leftover amount. What is this extra amount of juice?
Melissa: At first I was thinking that it should be 3 1/4 because there was enough juice for 3 whole servings and then there was an extra 1/4 [gallon] leftover, but now I can see it's 1/3 of a serving, because you only need 3 pieces for a full serving.

Fig. 5 Given an expression including division of fractions, one group devised the following story and illustrated it with initial diagram (a); diagram (b) illustrates the solution.

The principal has 2 1/2 gallons of juice for the preschool classes. A 3/4 gallon serving will go to each teacher. How many servings are in the 2 1/2 gallons?



the **sidebar** above. Sending preservice teachers back to their small groups to consider why this might be a division problem encourages them to build on prior knowledge. Melissa's rephrasing of the question as "How many 3/4 gallon servings are in 2 1/2 gallons?" helps remind David of measurement division in a whole-number context. Asking them to reason about the solution, including the initial erroneous remainder, causes them to revisit and revise their earlier thinking. The group eventually reaches a consensus on how to interpret the remainder by referring to the appropriate referent unit of the quotient (1/3 serving rather than 1/4 gallon).

Although it is premature to summarize for our learners the connections between actions on their diagrams and the steps in the invert-and-multiply algorithm, we provide an aside for the reader to make those connections explicit. The original 2 1/2 gallons is equivalent to 10 quarter gallons, the 10 arising from 2 1/2 × 4. Then the 10 quarter gallons are measured out by 3s, reflecting division by 3 (see **fig. 5b**). The picture illustrates

$$\left(2\frac{1}{2}\times 4\right)\div 3,$$

which is equivalent to

$$2\frac{1}{2}\times\frac{4}{3},$$

the expression resulting from inverting the divisor and multiplying.

The instructor next requests a story that uses a partitive interpretation of the same expression:

$$2\frac{1}{2}\quad\frac{3}{4}$$

Writing these stories is highly challenging work. Many partitive stories written during the whole-number unit had involved sharing some number of items among a certain number of people. With a problem such as

$$2\frac{1}{2}\quad\frac{3}{4},$$

most learners are unsure how to handle 2 1/2 items "shared" among 3/4 of a person. But because interpreting operations in several different ways is highlighted throughout the course, one group's effort to write a different story is tentatively presented to the rest of the class.

Students were then asked to consider the question in **figure 6**. The instructor does not attempt to sort out which interpretation of division it is but waits until learners have a chance to grapple with solving the problem. (See the **sidebar,** "Student Discussion of Figure 6's Driving Question," for the verbatim dialogue.) Rana's explanation helps the group recognize that since 2 1/2 miles represent 3/4 of the way, it makes sense to divide 2 1/2 miles by 3 to find 1/4 of the way.

Then students turn to the nontrivial work of finding 1/4 of the way using the diagram. One method involves partitioning each mile into thirds and then finding 1/3 of the remaining 1/2 mile. They reason that 1/4 of the way must be the following fraction of a mile:

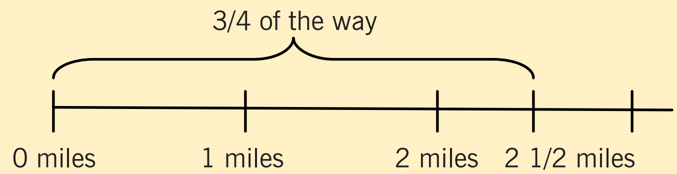
$$\frac{1}{3}+\frac{1}{3}+\frac{1}{6}=\frac{5}{6}$$

In this partitive case (see the **sidebar**, "Student Discussion," part 2), to find "one quarter of the way" when 3/4 is known, the original 2 1/2 mile distance was divided into 3 equal parts. The resulting value was multiplied by 4 to find the whole, or 4/4. Essentially, the solution embodies the calculation

Fig. 6 A partitive story is also accompanied by a diagram.

So far, I have driven 2 1/2 miles to get to my sister's house, but that distance is only 3/4 of the way.

How many miles would the total distance be?



Student Discussion of Figure 6's Driving Question

PART 1

Teacher: How would you solve this story problem?

Megan: I think we need to find that extra 1/4 and add it on. So maybe just find a fourth of 2 1/2?

Nicole: Yeah, but you want a quarter of the whole way, and since 2 1/2 miles is 3/4 of that, I think you just divide 2 1/2 by 3 to find a quarter [because the 2 1/2 comprises 3/4 of the distance].

Megan: Why divide by 3? Shouldn't we divide by 4 if we're trying to find a quarter?

Nicole: You need to know how many miles are in that quarter of the way, but since we have 3 of the quarters. . . . I don't know.

Teacher: Who can help us figure out why we divide by 3 and not 4?

Rana: When you divide the 3/4 [of the way] by 3, then you find what 1/4 of the way is.

PART 2

Teacher: Now that we know 1/4 of the way is 5/6 of a mile, some of you said to add that onto 2 1/2 miles to get the total distance. How else could you find the total distance?

Rana: When you have 1/4 of the way and you want 4/4 of the way, you could just multiply by 4.

Teacher: Let's summarize the steps we did with numbers. First, we divided 2 1/2 miles by 3. That gave us how far 1/4 of the way was. Then we took that result and multiplied by 4 to give us the whole distance to my sister's house. [Writes on board:

$$\left(2\frac{1}{2}\div 3\right)\times 4].$$

MEHMET ALI CIDA/ISTOCKPHOTO.COM

$$\left(2\frac{1}{2}\quad 3\right)\times 4,$$

which is, again, equivalent to

$$\left(2\frac{1}{2}\times 4\right)\quad 3.$$

The measurement and partitive stories each reveal the two steps of multiplication by 4/3 but in a different order. In the measurement illustration, 2 1/2 gallons was first *multiplied by 4*, resulting in 10 quarter gallons. These 10 quarter gallons were *divided by 3* to find how many 3/4 gallon servings were in 2 1/2 gallons. In the partitive case, 2 1/2 miles represented 3/4 of the way and so was *divided by 3* to find 1/4 of the way. The resulting distance (5/6 mile) was then *multiplied by 4* to find the distance of the entire trip. By highlighting and summarizing ideas, the instructor supports learners' understanding of multiplication by a fraction (a/b) as equivalent to two steps (accomplished in either order): divide by b and multiply by a.

ENACTING TASKS

Teachers' making sense of mathematical ideas for themselves is a prerequisite to teaching students for understanding (Ball and Bass 2000; Ball, Hill, and Bass 2005). The tasks described here provide preservice teachers with opportunities to understand why fraction division works the way it does. We expect some aspects to transfer directly to middle grades classrooms. For example, problems that require students to consider both division interpretations, attend to appropriate referent units, and form connections among representations promote a foundation for understanding fraction division.

Engaging with challenging tasks is key to learning mathematics with understanding. Equally important is the way that tasks are set up and implemented to maintain their cognitive demand (Stein et. al 2009). Providing time to think, highlighting big ideas, pressing for reasoning, and emphasizing connections are some instructional actions that support adult learners in our classrooms. These elements can be applied as well to implementing similar tasks in classrooms (Cengiz, Kline, and Grant 2011; Fraivillig 2001).

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